Complex numbers

Complex numbers are an expansion of a set of real numbers. There is no solution in real numbers for the equation

$$x^2 + 1 = 0$$

Therefore the imaginary number **i** is defined as

$$i^2 = -1$$

For the complex number z = x + **i**y we can say that x is a real part of a complex number and y is an imaginary part, and write it as

$$x = \mathrm{Re}(z) \qquad y = \mathrm{Im}(z)$$

Complex numbers such as -5 +0i, 9 - 0i have real numbers which are -5 and 9. Numbers 0 + i, 0 - 2i we call pure imaginary numbers.

Elementary operations

Elementary operations of complex numbers are addition, subtraction, multiplication and division:

- addition

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

for example

$$(2 + 3i) + (4 + 7i) = 6 + 10i$$

- subtraction

$$(a + ib) - (c + id) = (a - c) + i(b - d)$$

for example

$$(2 + 3i) - (4 + 7i) = -2 - 4i$$

- multiplication

$$(a + ib) \cdot (c + id) = ac + i^2 bd + ibc + iad = (ac - bd) + i(bc + ad)$$

for example

$$(2 + 3i) \cdot (4 + 7i) = -13 + 26i$$

- division

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id} \cdot \frac{c-id}{c-id} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

for example

$$(2+3i)/(4+7i) = 0.446 - 0.031i$$

Conjugation

If we have the complex number, z = x + iy, conjugate a complex number to a given number is z = x - iy. Conjugation of complex numbers is the function

$$z \to \mathrm{conj}(z) : \; C \to C$$

with the following properties

$$\mathrm{conj}(z_1 + z_2) = \mathrm{conj}(z_1) + \mathrm{conj}(z_2)$$

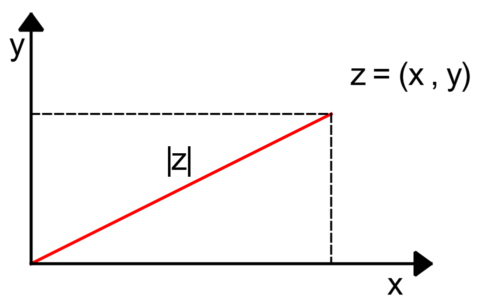
$$\mathrm{conj}(z_1 \cdot z_2) = \mathrm{conj}(z_1) \cdot \mathrm{conj}(z_2)$$

$$\mathrm{conj}\left(\frac{z_1}{z_2}\right) = \frac{\mathrm{conj}(z_1)}{\mathrm{conj}(z_2)}$$

$$\mathrm{conj}\big(\mathrm{conj}(z_1)\big) = z_1$$

Absolute value of complex number

Complex numbers can be presented graphically



Distance of the point from the origin to point z is an absolute value of a complex number and notation is |z|.

$$|z| = \sqrt[2]{x^2 + y^2}$$

Absolute value of a complex number is always a nonnegative number.
Some of the properties of absolute value of complex numbers are:

$$|x| < |z| \qquad\qquad |y| < |z|$$

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

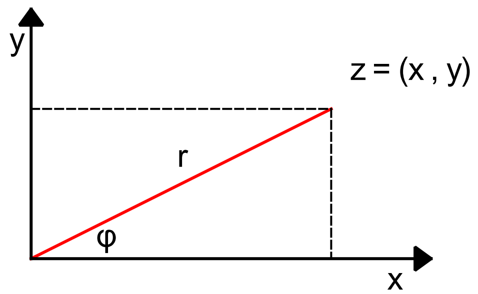
$$|z_1 + z_2| <= |z_1| + |z_2|$$

$$|\mathrm{conj}(z)| = |z|$$

$$|z|^2 = z \cdot \mathrm{conj}(z)$$

Complex number polar form

We can present complex numbers in a different form



In polar coordinates we have

$$x = r \cdot \cos(\varphi) \qquad r \geq 0, \quad 0 \leq \varphi < 2\pi$$
$$y = r \cdot \sin(\varphi)$$

So the complex number can be written in polar coordinates as below

$$z = x + iy = r\left(\cos(\varphi) + i\ \sin(\varphi)\right)$$

This formula as a consequence has Moaver's theorem

$$z^n = r^n\left(\cos(n\varphi) + i\ \sin(n\varphi)\right)$$

N-th root of complex number

Number ψ is a n-th root of a complex number z = x + iy if

$$\psi^{n} = z$$

If we use Moaver's theorem, for complex number z = r (cos(φ) + i sin(φ)) and number n from N we have

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \cdot \left(\cos\left(\frac{\varphi}{n}\right) + i\ \sin\left(\frac{\varphi}{n}\right)\right)$$

This will be the one solution for the problem we have started with. All solutions can be written in following form

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \cdot \left(\cos\left(\frac{\varphi + 2\ k\ \pi}{n}\right) + i\ \sin\left(\frac{\varphi + 2\ k\ \pi}{n}\right)\right) \qquad 0 \le \varphi < 2\pi,\ \ k = \ 0,1,...,n-1$$