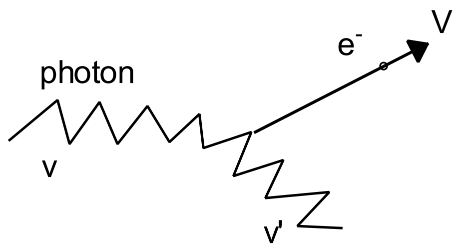
X-ray

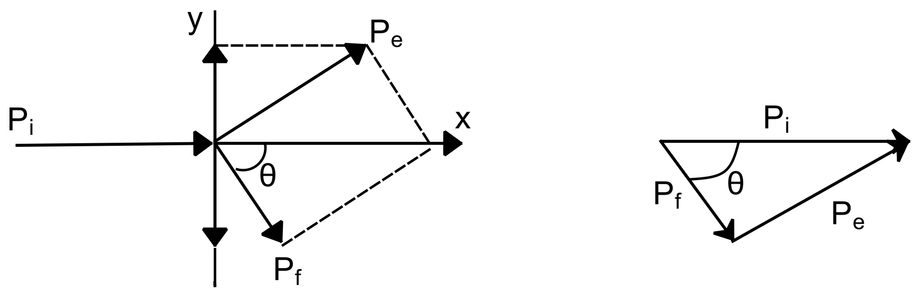
A 100-keV X ray is Compton scattered through an angle of 90°. What is the energy of the X ray after scattering?

Solution:

When X-rays pass through a crystal, some of the X-rays knock electrons out of the crystal atoms. The scattering of electromagnetic waves by the electrons in solids can be considered to be a collision between a photon of energy, hv, and an electron (shown on the Figure under).



The electron absorbs some of the energy of the incident photon and the scattered photon, so it has less energy and lower frequency v'. The collision is elastic, hence total energy as well as total momentum is conserved during the collision



$$E_i^{ph} + E_i^{el} = E_f^{ph} + E_f^{el}$$

$$P_i^{ph} = P_f^{ph} + P_f^{el}$$

The expressions for energies and moments are

$$P_i := \frac{hv}{c} \qquad P_f := \frac{hv'}{c} \qquad P_e := \frac{m_e v}{\sqrt[2]{1-\beta^2}}$$

$$E_i^{ph} := Pi\ c \qquad E_f^{ph} := P_f\ c \qquad E_i^{el} := m_e\ c^2 \qquad E_f^{el} := \frac{m_e\ c^2}{\sqrt[2]{1-\beta^2}}$$

where m_e is the rest mass of the electron, $\beta = v / c$. Conservation of momentum is shown schematically in the above figures. Using the law of cosines for the triangle in the figure below, adding and subtracting the quantity, $2p_ip_f$, on the right hand side, we get

$$P_e^2 = P_i^2 + P_f^2 - 2\,P_i\,P_f \cos\theta$$

$$P_e^2 = (P_i - P_f)^2 + 2\,P_i\,P_f\,(1 - \cos\theta)$$

The expression for the electronic energy is modified to eliminate β from the expression for our convenience.

$$\left(E_f^{el}\right)^2 = \left(-\beta^2\ m_e^{-2}\ c^{-4} + m_e^{-2}\ c^{-4}\right)^{-1}$$

$$\left(E_f^{el}\right)^2 - m_e^{2} \cdot c^4 = \left(-\beta^2\ m_e^{-2}\ c^{-4} + m_e^{-2}\ c^{-4}\right)^{-1} - c^4\ m_e^{2}$$

$$(E_f^{el})^2 = m_e^2\ c^4 + P_e^2\,c^2$$

From the conservation of energy

$$(m_e^2\,c^4 + P_e^2\,c^2)^{0.5} = (P_i - P_f)\,c + m_e\,c^2$$

Squaring both sides, we obtain another expression for P_e^2:

$$P_e^2 = (P_i - P_f)^2 + 2m_ec(P_i - P_f)$$

Equating the two expressions we obtained for P_e^2:

$$(P_i - P_f)^2 + 2 P_i P_f (1 - \cos\theta) = (P_i - P_f)^2 + 2 m_e c (P_i - P_f)$$

$$P_i P_f (1 - \cos\theta) = m_e c (P_i - P_f)$$

We now substitute (hv / c) and (hv' / c) for P_i and P_f respectively.

$$v\, v' (1 - \cos\theta) = (m_e c^2) / h\ (v - v')\text{eq}$$

Dividing both sides by v v' we will obtain Compton's equation

$$\frac{1}{hv'} = \frac{1}{hv} + \frac{1}{m_e c^2} \cdot (1 - \cos(\theta))$$

$$m_e c^2 = 511\ \text{keV} \qquad \cos(90) = 0$$

$$hv := 100\ \text{keV} \qquad h = m_e\, c^2 \qquad h := 511\ \text{keV}$$

$$hv' := \frac{hv\, h}{hv + h} \cdot (1 - \cos(90))$$

$$hv' = 83.633\ \text{keV}$$

The electron carrier of the remainder of the incident energy:

$$KE := 100\ \text{keV} - hv'$$

$$KE = 16.367\ \text{keV}$$